MMAT5360 200330

Cluedo

There are $m+n+1$ distinct cards

Player I : dealt m of them

Player II : dealt n of them

Remaining card: target card

Goal : guess what the target card is

On each turn, a player may : Guess or Ask

Guess : payoff $=1$ if correct

payoff $=0$ if wrong

$G_{m,n}$ Let $V_{m,n} = v(G_{m,n})$

$$\begin{cases} V_{m,0} = 1 , & m \geq 0 \\ V_{0,n} = \dfrac{1}{n+1} , & n \geq 0 \end{cases}$$

If Player I Asks a card he does not have (honest strategy), then play the game

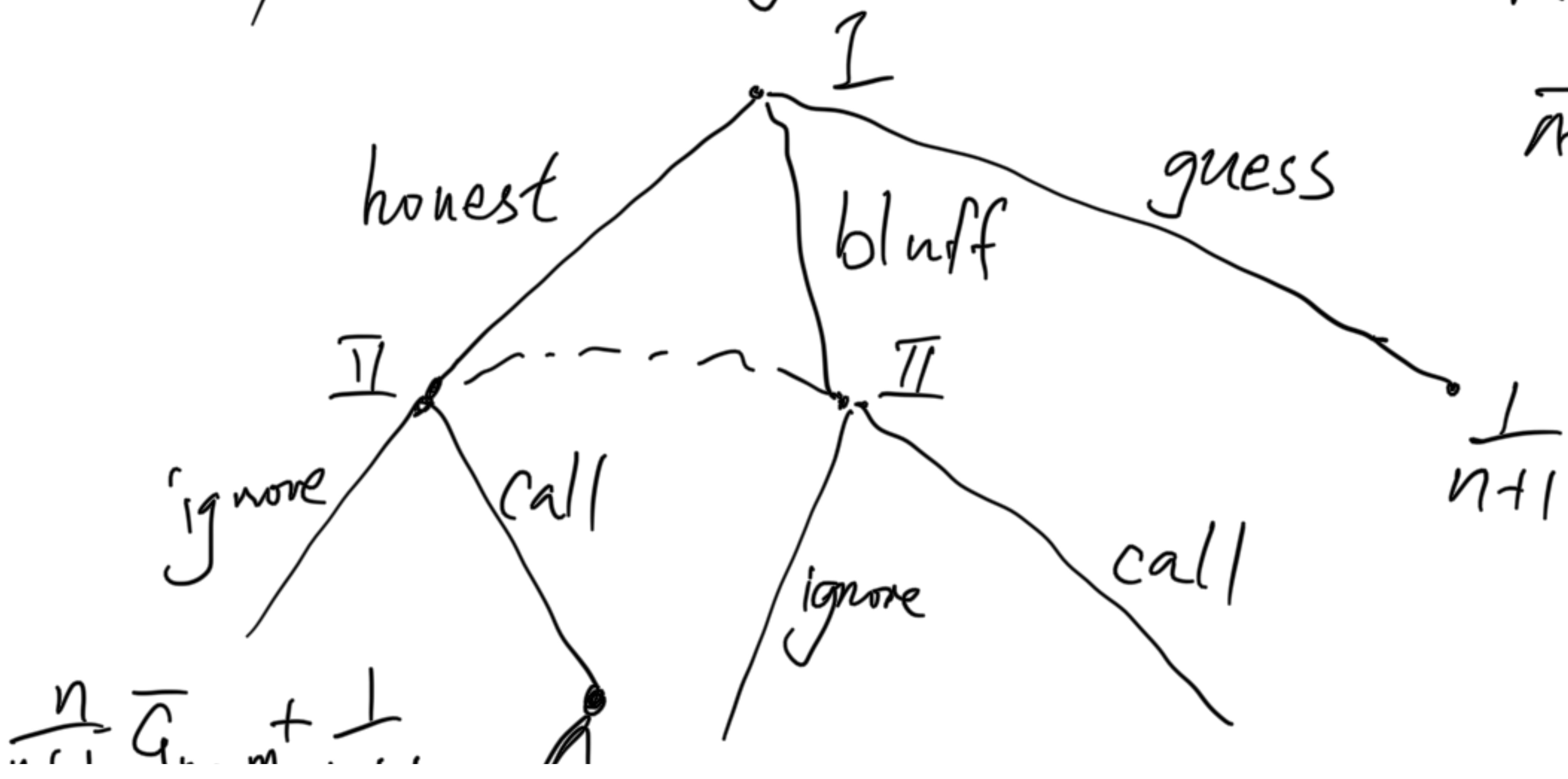
$\bar{G}_{n-1,m}$: Player II holds $n-1$ and moves first

$$v(\bar{G}_{n-1,m}) = 1 - v(G_{n,m})$$

Player I : Honest, Bluff, Guess

player II : Ignore, Call (guess the card player I asked)

$\frac{1}{n+1} \times 1 + \frac{n}{n+1} \times 0$



$$\bar{G}_{n,m-1} \qquad\qquad 1$$

$$\frac{n}{n+1}\bar{G}_{n-1,m}$$

	Ignore	Call
Honest	$\frac{n}{n+1}\bar{G}_{n-1,m}+\frac{1}{n+1}$	$\frac{n}{n+1}\bar{G}_{n-1,m}$
Bluff	$\bar{G}_{n,m-1}$	1
Guess	$\frac{1}{n+1}$	$\frac{1}{n+1}$

Guess : dominated strategy

Reduces to

$$\begin{pmatrix} \frac{n}{n+1}\bar{G}_{n-1,m}+\frac{1}{n+1} & \frac{n}{n+1}\bar{G}_{n-1,m} \\ \bar{G}_{n,m-1} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{n}{n+1}(1 - V_{n-1,m}) + \frac{1}{n+1} & \frac{n}{n+1}(1 - V_{n-1,m}) \\ 1 - V_{n,m-1} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{n+1-nV_{n-1,m}}{n+1} & \frac{n-nV_{n-1,m}}{n+1} \\ 1 - V_{n,m-1} & 1 \end{pmatrix}$$

Recall: $\begin{cases} V_{m,0} = 1 \\ V_{0,n} = \frac{1}{n+1} \end{cases}$

$$V_{1,1} = v \begin{pmatrix} \frac{2 - V_{0,1}}{2} & \frac{1 - V_{0,1}}{2} \\ 1 - V_{1,0} & 1 \end{pmatrix}$$

$$= v \begin{pmatrix} 3/4 & 1/4 \\ 0 & 1 \end{pmatrix} = \frac{1}{2}$$

$$V \quad = v \Big(\frac{2 - V_{0,2}}{} \quad \frac{1 - V_{0,2}}{} \Big)$$

$$v_{2,1} = v\begin{pmatrix} \frac{\;}{2} & \frac{\;}{2} \\ 1 - v_{1,1} & 1 \end{pmatrix}$$

$$= v\begin{pmatrix} 5/6 & 1/3 \\ 1/2 & 1 \end{pmatrix}$$

$$= \frac{2}{3}$$

N-person Cooperative games

Transferable utility (TU) games

Let $\mathcal{A} = \{A_1, A_2, \cdots, A_n\}$ be set of players.

Let X_i, $i = 1, 2, \cdots, n$: set of strategies of A_i

Payoff function :

Payoff function:

$$\vec{\pi} = (\pi_1, \pi_2, \cdots, \pi_n) : X_1 \times X_2 \times \cdots \times X_n \longrightarrow \mathbb{R}^n$$

π_i is the payoff of A_i

Def. $S \subset \mathcal{A}$: coalition

ϕ : empty coalition

$\mathcal{A}$: grand coalition

The total number coalitions is 2^n

$S^c = \mathcal{A} \setminus S$: counter coalition

Characteristic function:

$$\nu : \mathcal{P}(\mathcal{A}) \longrightarrow \mathbb{R}$$

power set of A

set of all subset of A

For $S \subset \mathcal{A}$, $v(S)$ is defined as the maximin value of S for the 2-person non-cooperative game between S and S^c.

Example: $\mathcal{A} = \{A_1, A_2, A_3\}$, $X_i = \{1, 2\}$

Strategies	$\vec{\pi}$
$(1, 1, 1)$	$(-2, 1, 2)$
$(1, 1, 2)$	$(1, 1, -1)$
$(1, 2, 1)$	$(0, -1, 2)$
$(1, 2, 2)$	$(-1, 2, 0)$
$(2, 1, 1)$	$(1, -1, 1)$
$(2, 1, 2)$	$(0, 0, 1)$
$(2, 2, 1)$	$(1, 0, 0)$
$(2, 2, 2)$	$(1, 2, -2)$

(A, B)

S $v(A)$

S^c $v(B^T)$

For $S = \{A_1, A_2\}$

For $S = \{A_1, A_2\}$,

$S^c = \{A_3\}$

		S^c: 1	2
S	$(1,1)$	$(-1,2)$	$(2,-1)$
	$(1,2)$	$(-1,2)$	$(1,0)$
	$(2,1)$	$(0,1)$	$(0,1)$
	$(2,2)$	$(1,0)$	$(3,-2)$

$$v(S) = v\begin{pmatrix} -1 & 2 \\ -1 & 1 \\ 0 & 0 \\ 1 & 3 \end{pmatrix} = 1$$

$$v(S^c) = v\begin{pmatrix} 2 & 2 & 1 & 0 \\ -1 & 0 & 1 & -2 \end{pmatrix} = 0$$

Constant sum game

S $\qquad\qquad$ $v(S)$

$\{A_1\}$	$\frac{[\text{illegible}]}{4}$
$\{A_2\}$	$-\frac{1}{3}$
$\{A_3\}$	0
$\{A_1, A_2\}$	1
$\{A_1, A_3\}$	$\frac{4}{3}$
$\{A_2, A_3\}$	$\frac{3}{4}$
$\{A_1, A_2, A_3\}$	1
$\emptyset$	0

Example: Used car game

	value the car
A_1	0
A_2	$\$500$

S	$v(S)$
$\{A_1\}, \{A_2\}, \{A_3\}$	0
$\{A_1, A_2\}$	500

A_3 \$700

$\{A_1, A_2\}$	[illegible]
$\{A_1, A_3\}$	700
$\{A_2, A_3\}$	0
$\{A_1, A_2, A_3\}$	700
$\emptyset$	0

Thm: (Superadditivity)

For any $S, T \subset \mathcal{A}$ with $S \cap T = \emptyset$,

$$v(S \cup T) \geq v(S) + v(T)$$

If $S_1, S_2, \ldots, S_k$ are pairwise disjoint coalitions then

$$v\left(\bigcup_{i=1}^{k} S_i\right) \geq \sum_{i=1}^{k} v(S_i)$$

In particular, $v(\mathcal{A}) \geq \sum^{n} v(\{A_i\})$

$$v(\mathcal{A}) \geq \sum_{i=1} v(\{A_i\})$$

Def. The characteristic form of a game is a function $v : \mathcal{P}(\mathcal{A}) \to \mathbb{R}$ s.t.

1. $v(\emptyset) = 0$
2. (Superadditivity) If $S, T \in \mathcal{P}(\mathcal{A})$ with $S \cap T = \emptyset$ then $v(S \cup T) \geq v(S) + v(T)$.

Def. A game $(\mathcal{A}, v)$ is <u>essential</u> if

$$v(\mathcal{A}) > \sum_{i=1}^{n} v(\{A_i\})$$

$(\mathcal{A}, v)$ is inessential if

$$v(\mathcal{A}) = \sum_{i=1}^{n} v(\{A_i\})$$

Thm: If $(\mathcal{A}, \nu)$ is inessential, then for and $S \subset \mathcal{A}$, we have

$$\nu(S) = \sum_{A_i \in S} \nu(\{A_i\})$$

Proof. By superadditivity

$$\nu(S) \geq \sum_{A_i \in S} \nu(\{A_i\}) \text{ and } \nu(S^c) \geq \sum_{A_j \in S^c} \nu(\{A_j\})$$

If $(\mathcal{A}, \nu)$ is inessential,

$$\begin{aligned}\nu(\mathcal{A}) = \sum_{i=1}^{n} \nu(\{A_i\}) &= \sum_{A_i \in S} \nu(\{A_i\}) + \sum_{A_j \in S^c} \nu(\{A_j\}) \\ &\leq \nu(S) + \nu(S^c) \\ &\leq \nu(\mathcal{A})\end{aligned}$$

$\therefore \quad v(S) = \sum_{A_i \in S}' v(\{A_i\})$

Def An n-vector $\vec{x} \in (x_1, \cdots, x_n) \in \mathbb{R}^n$ is called an <u>imputation</u> for $(\mathcal{A}, v)$ if

1. $x_i \geq v(\{A_i\}) \quad i = 1, 2, \cdots, n$
2. $\sum_{i=1}^{n} x_i = v(\mathcal{A})$

The set of imputations is denoted by $I(v)$

Note: v is inessential $\Rightarrow I(v) = \{(v(\{A_1\}), v(\{A_2\}), \cdots, v(\{A_n\}))\}$

v is essential $\Rightarrow$ There are infinitely many imputations

Core and Shapley values

Def. The core of $(\mathcal{A}, v)$ consists of

$\vec{x} \in I(v)$ such that for any $S \subset \mathcal{A}$,
we have

$$\sum_{A_i \in S} x_i \geq v(S)$$

$\vec{x} = (x_1, x_2, \cdots, x_n)$

The set of all imputations in the core is denoted by $C(v)$

$$C(v) \subset I(v)$$